MATH 2010 Advanced Calculus

Suggested Solution of Homework 8

Q2:

Solution

$f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$; $f_{xx}\left(\frac{2}{3}, \frac{4}{3}\right) = -10$, $f_{yy}\left(\frac{2}{3}, \frac{4}{3}\right) = -4$, $f_{xy}\left(\frac{2}{3}, \frac{4}{3}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$

Q11:

Solution

$f_x(x, y) = \frac{112x - 8x}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0$ and $f_y(x, y) = \frac{-8y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0 \Rightarrow$ critical point is $\left(\frac{16}{7}, 0\right)$; $f_{xx}\left(\frac{16}{7}, 0\right) = -\frac{8}{15}$, $f_{yy}\left(\frac{16}{7}, 0\right) = -\frac{8}{15}$, $f_{xy}\left(\frac{16}{7}, 0\right) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{64}{225} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{16}{7}, 0\right) = -\frac{16}{7}$

Q16:

Solution

$f_x(x, y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x, y) = 3x^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 2) = -12$; for $(-2, 0)$: $f_{xx}(-2, 0) = -6$, $f_{yy}(-2, 0) = -6$, $f_{xy}(-2, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, 0) = -4$; for $(-2, 2)$: $f_{xx}(-2, 2) = -6$, $f_{yy}(-2, 2) = 6$, $f_{xy}(-2, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

Q20:

Solution

$f_x(x, y) = 4x^3 + 4y = 0$ and $f_y(x, y) = 4y^3 + 4x = 0 \Rightarrow x = -y \Rightarrow -x^3 + x = 0 \Rightarrow x\left(1-x^2\right) = 0 \Rightarrow x = 0, 1, -1$
$\Rightarrow$ the critical points are $(0, 0), (1, -1),$ and $(-1, 1)$; $f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$, and $f_{xy}(x, y) = 4$;
for $(0, 0)$: $f_{xx}(0, 0) = 0, f_{yy}(0, 0) = 0, f_{xy}(0, 0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1, -1)$:
$f_{xx}(1, -1) = 12, f_{yy}(1, -1) = 12, f_{xy}(1, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of
$f(1, -1) = -2$; for $(-1, 1)$: $f_{xx}(-1, 1) = 12, f_{yy}(-1, 1) = 12, f_{xy}(-1, 1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and
$f_{xx} > 0 \Rightarrow$ local minimum of $f(-1, 1) = -2$

Q22:

Solution

$f_x(x, y) = -\frac{1}{x^2} + y = 0$ and $f_y(x, y) = x - \frac{1}{y^2} = 0 \Rightarrow x = 1 \Rightarrow$ and $y = 1$ the critical point is $(1, 1)$;
$f_{xx} = \frac{2}{x^3}, f_{yy} = \frac{2}{y^3}, f_{xy} = 1$; $f_{xx}(1, 1) = 2, f_{yy}(1, 1) = 2, f_{xy}(1, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 2$
$\Rightarrow$ local minimum of $f(1, 1) = 3$

Q24:

Solution

$f_x(x, y) = 2e^{2x}\cos y = 0$ and $f_y(x, y) = -e^{2x}\sin y = 0 \Rightarrow$ no solution since $e^{2x} \neq 0$ for any x and the functions $\cos y$ and $\sin y$ cannot equal 0 for the same $y \Rightarrow$ no critical points $\Rightarrow$ no extrema and no saddle points

Q27:

Solution

$f_x(x, y) = 2xe^{-y} = 0$ and $f_y(x, y) = 2ye^{-y} - e^{-y}\left(x^2 + y^2\right) = 0 \Rightarrow$ critical points are $(0, 0)$ and $(0, 2)$;
for $(0, 0)$: $f_{xx}(0, 0) = 2e^{-y}\Big|_{(0,0)} = 2, f_{yy}(0, 0) = \left(2e^{-y} - 4ye^{-y} + e^{-y}\left(x^2 + y^2\right)\right)\Big|_{(0,0)} = 2,$
$f_{xy}(0, 0) = -2xe^{-y}|_{(0,0)} = 0, \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0 \Rightarrow$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 0) = 0$;
for $(0, 2)$: $f_{xx}(0, 2) = 2e^{-y}\Big|_{(0,2)} = \frac{2}{e^2}, f_{yy}(0, 2) = \left(2e^{-y} - 4ye^{-y} + e^{-y}\left(x^2 + y^2\right)\right)\Big|_{(0,2)} = -\frac{2}{e^2},$
$f_{xy}(0, 2) = -2xe^{-y}\Big|_{(0,2)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -\frac{4}{e^4} < 0 \Rightarrow$ saddle point

Q30:

Solution

$f_x(x, y) = 2x + \frac{1}{x+y} = 0$ and $f_y(x, y) = -1 + \frac{1}{x+y} = 0 \Rightarrow$ critical point is $\left(-\frac{1}{2}, \frac{3}{2}\right)$; $f_{xx}\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$, $f_{yy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1$, $f_{xy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -2 < 0 \Rightarrow$ saddle point

Q44(d):

Solution

Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$

Q67:

Solution

$$m = \frac{(2)(-1) - 3(-14)}{(2)^2 - 3(10)} = -\frac{20}{13} \text{ and}$$

$$b = \frac{1}{3}\left[-1 - \left(-\frac{20}{13}\right)(2)\right] = \frac{9}{13}$$

$$\Rightarrow y = -\frac{20}{13}x + \frac{9}{13}; y\big|_{x=4} = -\frac{71}{13}$$

Solution